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A new cubical h-vector ☆

Ron M. Adin *

Department of Mathematics and Computer Science, Bar-Ilan University, Ramat-Gan 52900, Israel

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Abstract

A new definition of an h-vector for cubical polytopes (and complexes) is introduced. It has many properties in common with the well-known h-vector for simplicial polytopes. In particular, it is symmetric, nonnegative and easily computable from a shelling of the polytope. Lower or upper bounds on its components imply corresponding bounds on the face numbers.

Résumé

On introduit un nouveau vecteur h défini pour des polytopes (ainsi que pour des complexes) cubiques. Celui-ci possède de nombreuses propriétés enjouies par le vecteur h habituel des polytopes simpliciaux. Notamment, ce nouveau vecteur h est symétrique et positif et se calcule facilement à partir d'un effeuillage du polytope. Des bornes inférieures et supérieures pour ses composantes entrainent des bornes pour le nombres de faces.

1. Introduction

A *d-polytope* P is the convex hull of finitely many points which affinely span $\mathbb{R}^d$. The intersections of P with its various supporting hyperplanes are its (proper) *faces*, and are called *vertices*, *edges* or *facets* if they are of dimension 0, 1 or $d-1$, respectively. (The improper faces are P itself and the empty set.) A d-polytope is *cubical* if all its proper faces (equivalently: all its facets) are combinatorially equivalent to cubes (respectively, to $(d-1)$-cubes).

More generally, let C^{d-1} be the standard cube $[0,1]^{d-1}$ in $\mathbb{R}^{d-1}$, and let $V_0 = \operatorname{vert} C^{d-1}$ be its set of vertices. A (finite, pure, abstract) *cubical* $(d-1)$*-complex* consists of a finite nonempty set V together with a (finite) nonempty collection $\{\phi_\alpha\}_{\alpha\in I}$ of distinct injective maps $\phi_\alpha : V_0 \to V$, such that $\phi_\alpha^{-1}(\phi_\beta(V_0))$ is the set of vertices of

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* E-mail: radin@bimacs.cs.biu.ac.il.

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a (proper or improper) face of C^{d-1}, for all $\alpha, \beta \in I$. The images (under the various maps ϕ_α) of i-dimensional faces of C^{d-1} $(0 \leqslant i \leqslant d-1)$ are the *i-faces* (or *i-cubes*) of the complex. The (vertex sets of) facets of a cubical d-polytope form a cubical $(d-1)$-complex.

Now let K be a cubical $(d-1)$-complex, and let f_i be the number of i-cubes in it $(0 \leqslant i \leqslant d-1)$. The vector $(f_0, \ldots, f_{d-1})$ is known as the *f-vector* of K. Define a *short cubical h-vector* $(h_0^{(sc)}, \ldots, h_{d-1}^{(sc)})$ and a corresponding *short cubical h-polynomial* $h_K^{(sc)}(q)$ for the complex K by

$$h_K^{(sc)}(q) = \sum_{i=0}^{d-1} h_i^{(sc)} q^i \overset{\text{def}}{=} \sum_{j=0}^{d-1} f_j (2q)^j (1-q)^{d-1-j}. \tag{1}$$

Define also a (long) *cubical h-vector* $(h_0^{(c)}, \ldots, h_d^{(c)})$ by the recursive formula

$$h_i^{(sc)} = h_i^{(c)} + h_{i+1}^{(c)} \quad (0 \leqslant i \leqslant d-1), \tag{2}$$

together with the initial value

$$h_0^{(c)} = 2^{d-1}. \tag{3}$$

Equivalently, the *cubical h-polynomial* $h_K^{(c)}(q)$ may be defined by

$$h_K^{(c)}(q) = \sum_{i=0}^{d} h_i^{(c)} q^i \overset{\text{def}}{=} \sum_{j=0}^{d} f_{j-1} c_j(q), \tag{4}$$

where the convention $f_{-1} = 1$ is used (to account for the empty set), and where the polynomials $c_j(q)$ are defined by

$$(1+q)c_j(q) = \begin{cases} 2^{d-1} \cdot [1-(-q)^{d+1}] & \text{if } j = 0; \\ 2^{j-1} q^j (1-q)^{d-j} + (-1)^{d-j} 2^{d-1} q^{d+1} & \text{if } 1 \leqslant j \leqslant d. \end{cases} \tag{5}$$

These definitions are reminiscent of the defining equation for the h-vector of a *simplicial* $(d-1)$-complex Σ, which is

$$h_\Sigma^{(s)}(q) = \sum_{i=0}^{d} h_i^{(s)} q^i \overset{\text{def}}{=} \sum_{j=0}^{d} f_{j-1} q^j (1-q)^{d-j}. \tag{6}$$

The simplicial h-vector has some very appealing properties, and has been found to be an invaluable tool in the formulation and proof of results in the enumerative theory of simplicial convex polytopes. Typical examples are the proof by McMullen [9] of Motzkin's upper bound conjecture [11], and the complete characterization result ("McMullen's g-conjecture") [10] proved by Stanley [12] (necessity) and Billera and Lee [1] (sufficiency). The cubical h-vector introduced above shares some of these properties, and will hopefully find appropriate use in the (recently reviving) study of cubical polytopes.

2. Properties of the cubical *h*-vector

Let us first collect a few immediate observations.

Lemma 1. *Let K be a cubical $(d-1)$-complex. Then:*

(i) *All the numbers $h_i^{(sc)}$ $(0 \leqslant i \leqslant d-1)$ are integers.*

(ii)

$$h_K^{(sc)}(0) = h_0^{(sc)} = f_0, \tag{7}$$

$$h_K^{(sc)}(1) = \sum_{i=0}^{d-1} h_i^{(sc)} = 2^{d-1} f_{d-1} \tag{8}$$

and

$$h_K^{(sc)}(-1) = \sum_{i=0}^{d-1} (-1)^i h_i^{(sc)} = 2^{d-1}\chi(K), \tag{9}$$

where

$$\chi(K) = \sum_{j=0}^{d-1} (-1)^j f_j(K) \tag{10}$$

is the Euler characteristic of K.

(iii) *Explicitly,*

$$h_i^{(sc)} = \sum_{j=0}^{i} \binom{d-1-j}{d-1-i} (-1)^{i-j} 2^j f_j \quad (0 \leqslant i \leqslant d-1) \tag{11}$$

and

$$f_j = 2^{-j} \sum_{i=0}^{j} \binom{d-1-i}{d-1-j} h_i^{(sc)} \quad (0 \leqslant j \leqslant d-1). \tag{12}$$

(iv) *For the boundary complex of the d-cube,*

$$h_0^{(sc)} = \cdots = h_{d-1}^{(sc)} = 2^d. \tag{13}$$

Analogous properties hold for the (long) cubical h-vector, as follows.

Lemma 2. *Let K be a cubical $(d-1)$-complex. Then:*

(i) *All the numbers $h_i^{(c)}$ $(0 \leqslant i \leqslant d)$ are integers.*

(ii)

$$h_0^{(c)} = 2^{d-1}, \tag{14}$$

$$h_1^{(c)} = f_0 - 2^{d-1} \tag{15}$$

and

$$h_d^{(c)} = (-2)^{d-1}\tilde{\chi}(K), \tag{16}$$

where

$$\tilde{\chi}(K) = \sum_{j=0}^{d}(-1)^{j-1} f_{j-1}(K) \tag{17}$$

is the reduced Euler characteristic of K.

(iii) *More generally,*

$$h_i^{(c)} = (-1)^i 2^{d-1} f_{-1} + \sum_{j=1}^{i}(-1)^{i-j}2^{j-1}\left(\sum_{k=j}^{i}\binom{d-j}{k-j}\right) f_{j-1} \quad (1 \leqslant i \leqslant d) \tag{18}$$

and

$$f_{j-1} = 2^{1-j}\sum_{i=1}^{j}\binom{d-i}{d-j}\left[h_i^{(c)} + h_{i-1}^{(c)}\right] \quad (1 \leqslant j \leqslant d). \tag{19}$$

(iv) *For the boundary complex of the d-cube,*

$$h_0^{(c)} = \cdots = h_d^{(c)} = 2^{d-1}. \tag{20}$$

For comparison, here are the corresponding properties of the simplicial h-vector.

Lemma 3. *Let Σ be a simplicial $(d-1)$-complex. Then:*

(i) *All the numbers $h_i^{(s)}$ $(0 \leqslant i \leqslant d)$ are integers.*

(ii)

$$h_0^{(s)} = 1, \tag{21}$$

$$h_1^{(s)} = f_0 - d \tag{22}$$

and

$$h_d^{(s)} = (-1)^{d-1}\tilde{\chi}(\Sigma). \tag{23}$$

(iii) *More generally,*

$$h_i^{(s)} = \sum_{j=0}^{i}\binom{d-j}{d-i}(-1)^{i-j} f_j \quad (0 \leqslant i \leqslant d) \tag{24}$$

and

$$f_j = \sum_{i=0}^{j}\binom{d-i}{d-j} h_i^{(s)} \quad (0 \leqslant j \leqslant d). \tag{25}$$

(iv) *For the boundary complex of the d-simplex,*

$$h_0^{(s)} = \cdots = h_d^{(s)} = 1. \tag{26}$$

Proofs of these results follow from straightforward computations, and will be omitted. We want to emphasize just one important consequence of the fact that the transition coefficients in formulas (12) and (19) are nonnegative.

Corollary 4. *Lower or upper bounds on the cubical (or short cubical) h-numbers of a complex imply corresponding bounds on its f-numbers.*

We now want to state a few additional properties of the cubical h-numbers. For general terminology regarding partially ordered sets, refer to [13, Ch. 3].

Theorem 5. *Let K be a cubical $(d-1)$-complex, and denote by $\hat{P}(K)$ its lattice of faces augmented by a maximal element. Then:*

(i) *If $\hat{P}(K)$ is semi-Eulerian (e.g., if K is a cubical subdivision of a $(d-1)$-manifold without boundary) then its short cubical h-vector is symmetric:*

$$h_i^{(\mathrm{sc})} = h_{d-1-i}^{(\mathrm{sc})} \quad (0 \leqslant i \leqslant d-1). \tag{27}$$

(ii) *If $\hat{P}(K)$ is Eulerian (e.g., if K is a cubical subdivision of a $(d-1)$-sphere) then its cubical h-vector is symmetric:*

$$h_i^{(\mathrm{c})} = h_{d-i}^{(\mathrm{c})} \quad (0 \leqslant i \leqslant d). \tag{28}$$

Equivalently, its short cubical h-vector is symmetric and satisfies the additional equation

$$h_K^{(\mathrm{sc})}(-1) = \sum_{i=0}^{d-1}(-1)^i h_i^{(\mathrm{sc})} = 2^{d-1} + (-2)^{d-1}. \tag{29}$$

(iii) *Let $F_1,\ldots,F_m$ be a shelling of the facets of K. Define the t-th shelling step $(1 \leqslant t \leqslant m)$ to be of type (a_0,a_1,a_2) if, out of the $d-1$ antipodal pairs of subfacets (i.e., $(d-2)$-faces) in F_t, exactly a_i pairs $(i = 0,1,2)$ have i subfacets in common with the union $\bigcup_{s<t} F_s$ of preceding facets in the shelling. Necessarily $a_0+a_1+a_2 = d-1$, and either $a_1 = a_2 = 0$, $a_0 = a_1 = 0$ or $a_1 \geqslant 1$. Then*

$$h_K^{(\mathrm{sc})}(q) = \sum_{t=1}^{m} \Delta_t h_K^{(\mathrm{sc})}(q), \tag{30}$$

where the contribution of a shelling step of type (a_0,a_1,a_2) is

$$\Delta_t h_K^{(\mathrm{sc})}(q) = 2^{a_0}(1+q)^{a_1}(2q)^{a_2}. \tag{31}$$

Similarly,

$$h_K^{(\mathrm{c})}(q) = \sum_{t=1}^{m} \Delta_t h_K^{(\mathrm{c})}(q), \tag{32}$$

where the contribution of a shelling step of type (a_0,a_1,a_2) is

$$\Delta_t h_K^{(\mathrm{c})}(q) = \begin{cases} q \cdot 2^{a_0}(1+q)^{a_1-1}(2q)^{a_2} & \text{if } a_1 \geqslant 1, \\ 2^{d-1} & \text{if } (a_0,a_1,a_2) = (d-1,0,0), \\ 2^{d-1}q^d & \text{if } (a_0,a_1,a_2) = (0,0,d-1). \end{cases} \tag{33}$$

In particular, the cubical h-vector of a shellable cubical $(d-1)$-complex is nonnegative.

A proof of this theorem will be outlined in the following sections. Let us also restate, in cubical h-vector terminology, two recent results of G. Blind and R. Blind. The current restatement of Theorem 7 actually reflects part of its original proof.

Theorem 6 (Blind and Blind [2]). *A cubical (or, more generally, a triangle-free) d-polytope has at least 2^d vertices. Equivalently, for a cubical d-polytope*

$$h_0^{(c)} \leqslant h_1^{(c)}. \tag{34}$$

Theorem 7 (Blind and Blind [3]). *If P is a cubical polytope of even dimension $d \geqslant 4$ then any shelling of the facets of P contains an even number of steps of type $(0, d-1, 0)$. Therefore, all the numbers $h_i^{(c)}$ and $h_i^{(sc)}$ (and in particular $f_0 = h_0^{(sc)}$) are even.*

The conclusion of Theorem 7 is false for $d = 3$, and is still open for odd $d \geqslant 5$.

Sections 3 and 4 of this paper contain a proof of Theorem 5. Further remarks and some open problems are gathered in Section 5.

3. Generalized Dehn–Sommerville equations

In this section we shall prove part (ii) of Theorem 5 in the case that K is the boundary complex of a cubical d-polytope. As will be remarked later, essentially the same proof may be used to obtain parts (i) and (ii) in the general case as well. We outline the proof in the context of polytopes in order to reflect some of the ideas that led to the current definition of the short and long cubical h-vectors. Indeed, an aesthetic motivation for introducing definition (1), and subsequently also (2), has been the attempt to rephrase the cubical Dehn–Sommerville equations as a symmetry property like (28), in analogy with the simplicial case. A long-established reference on polytopes, including the generalized Dehn–Sommerville equations we are going to use, is [6].

Let P be an arbitrary convex d-polytope. The well-known *Euler relation* states that the reduced Euler characteristic of the boundary of P, which is homeomorphic to a $(d-1)$-sphere, is

$$\sum_{j=0}^{d}(-1)^{j-1} f_{j-1}(P) = (-1)^{d-1}. \tag{35}$$

Again, the convention $f_{-1} = 1$ is to be used here. For an arbitrary $(i-1)$-face F_{i-1} of P $(0 \leqslant i \leqslant d)$, the *link* $P/F_{i-1} = \mathrm{lk}_P(F_{i-1})$ has the structure of a convex $(d-i)$-polytope, so that (upon multiplying by $(-1)^{d-i-1}$)

$$\sum_{j=i}^{d}(-1)^{d-j} f_{j-i-1}(P/F_{i-1}) = 1. \tag{36}$$

Fixing i and summing over all the $(i-1)$-faces of P, one obtains the *generalized Dehn–Sommerville equations* for an arbitrary d-polytope P:

$$\sum_{j=i}^{d}(-1)^{d-j}f_{i-1,j-1}(P)=f_{i-1}(P) \quad (0\leqslant i\leqslant d). \tag{37}$$

Here $f_{i-1,j-1}$ is the number of flags $F_{i-1}\subseteq F_{j-1}$ where F_{i-1} (F_{j-1}) is an $(i-1)$-face (respectively, a $(j-1)$-face) of P.

If P is a *simplicial* d-polytope then

$$f_{i-1,j-1}(P)=\binom{j}{i}f_{j-1}(P) \quad (0\leqslant i\leqslant j\leqslant d) \tag{38}$$

and one obtains the *simplicial Dehn–Sommerville equations*:

$$\sum_{j=i}^{d}\binom{j}{i}(-1)^{d-j}f_{j-1}(P)=f_{i-1}(P) \quad (0\leqslant i\leqslant d). \tag{39}$$

Upon multiplying by $(q-1)^{d-i}$ and summing over all i, these are seen to be equivalent to the single polynomial equation

$$\sum_{j=0}^{d}f_{j-1}q^{j}(1-q)^{d-j}=\sum_{i=0}^{d}f_{i-1}(q-1)^{d-i}. \tag{40}$$

Using definition (6), this may be stated as

$$h^{(s)}(q)=q^{d}h^{(s)}(q^{-1}), \tag{41}$$

which amounts to the symmetry of the simplicial h-vector:

$$h_i^{(s)}=h_{d-i}^{(s)} \quad (0\leqslant i\leqslant d). \tag{42}$$

Analogously, let P be a *cubical* d-polytope. Then

$$f_{i-1,j-1}(P)=\begin{cases}\binom{j-1}{i-1}2^{j-i}f_{j-1}(P) & \text{if } 1\leqslant i\leqslant j\leqslant d,\\ f_{j-1}(P) & \text{if } 0=i\leqslant j\leqslant d,\end{cases} \tag{43}$$

and this implies (with a shift of indices) the *cubical Dehn–Sommerville equations*

$$\begin{aligned}&\sum_{j=i}^{d-1}\binom{j}{i}(-1)^{d-1-j}2^{j-i}f_j=f_i \ (0\leqslant i\leqslant d-1),\\ &\sum_{j=-1}^{d-1}(-1)^{d-1-j}f_j=f_{-1} \qquad (\text{Euler, } i=-1).\end{aligned} \tag{44}$$

The last equation is Euler's relation for P. The other d equations may be rewritten as

$$\sum_{j=i}^{d-1}\binom{j}{i}(-1)^{d-1-j}2^{j}f_j=2^{i}f_i \quad (0\leqslant i\leqslant d-1), \tag{45}$$

so that the vector $(f_0, 2f_1, 2^2 f_2, \ldots, 2^{d-1} f_{d-1})$ for a cubical d-polytope satisfies the same linear equations (39) as the vector $(f_{-1}, f_0, f_1, \ldots, f_{d-2})$ for a simplicial $(d-1)$-polytope. This motivates definition (1), leading to the rewriting of (44) as

$$\begin{cases} h^{(\mathrm{sc})}(q) = q^{d-1} h^{(\mathrm{sc})}(q^{-1}), \\ h^{(\mathrm{sc})}(-1) = 2^{d-1} + (-2)^{d-1} \quad \text{(Euler)} \end{cases} \tag{46}$$

or equivalently

$$\begin{cases} h_i^{(\mathrm{sc})} = h_{d-1-i}^{(\mathrm{sc})} & (0 \leqslant i \leqslant d-1), \\ \sum_{i=0}^{d-1} (-1)^i h_i^{(\mathrm{sc})} = 2^{d-1} + (-2)^{d-1} & \text{(Euler)}. \end{cases} \tag{47}$$

Definition (2) transforms these equations into

$$\begin{cases} h_i^{(\mathrm{c})} + h_{i+1}^{(\mathrm{c})} = h_{d-1-i}^{(\mathrm{c})} + h_{d-i}^{(\mathrm{c})} & (0 \leqslant i \leqslant d-1), \\ h_0^{(\mathrm{c})} + (-1)^{d-1} h_d^{(\mathrm{c})} = 2^{d-1} + (-2)^{d-1} & \text{(Euler)}, \end{cases} \tag{48}$$

and by (3) these are equivalent to

$$h_i^{(\mathrm{c})} = h_{d-i}^{(\mathrm{c})} \quad (0 \leqslant i \leqslant d). \tag{49}$$

This completes the proof of part (ii) of Theorem 5 for a complex K which is the boundary complex of a cubical d-polytope

Note. The only facts used in the proof which are not valid for arbitrary cubical $(d-1)$-complexes are the Euler relations (36) for the links of various faces in the polytope P. Similar equations, in fact, are satisfied by any Eulerian poset [13, p. 122], so the above reasoning also proves part (ii) in the general case. Even if $\hat{P}(K)$ is only semi-Eulerian, these equations still hold for the *nonempty* faces. Therefore all but the last equation in (44) are still valid, leading to a proof of part (i) of the theorem as well.

4. Shellability

In this section we shall prove part (iii) of Theorem 5.

A *shelling* of a (pure) cubical $(d-1)$-complex is an ordering $F_1, \ldots, F_m$ of its facets ($(d-1)$-faces) such that, for all $1 \leqslant t \leqslant m$, the intersection of F_t with the union $\bigcup_{s<t} F_s$ of preceding facets is a union of subfacets (i.e., $(d-2)$-faces) which is homeomorphic to either a $(d-2)$-ball, a $(d-2)$-sphere or (if $t = 1$) an empty set. If such a shelling exists, the complex K is called *shellable*. Shellability may be defined (in a more cautious manner) for more general cell complexes. A fundamental result in this area is the following.

Theorem 8 (Bruggesser and Mani [4]). *The boundary complex of every convex polytope is shellable.*

The proof of Theorem 7, for example, uses induction on a shelling of the polytope together with a clever coloring argument.

Fix a shelling $F_1, \ldots, F_m$ of a shellable cubical $(d-1)$-complex K. For a given step $1 \leqslant t \leqslant m$, call a face of F_t 'old' (respectively, 'new') if it is contained (is not contained) in $\bigcup_{s<t} F_s$. Recall (from the statement of Theorem 5, part (iii)) that the *type* of the t-th shelling step is the triple (a_0, a_1, a_2), where exactly a_i $(i = 0, 1, 2)$ out of the $d-1$ antipodal pairs of subfacets of F_t consist of i 'old' and $2-i$ 'new' subfacets. It is not difficult to see that the possible types either satisfy $a_1 \geqslant 1$ (if the union of 'old' subfacets is homeomorphic to a $(d-2)$-ball), or are equal to $(0, 0, d-1)$ (union homeomorphic to a $(d-2)$-sphere) or $(d-1, 0, 0)$ (empty union, for $t = 1$).

A j-face of F_t $(0 \leqslant j \leqslant d-2)$ is 'new' if and only if it is an intersection of $d-1-j$ 'new' subfacets, no two of which being antipodal. Enumeration of these 'new' faces gives

$$\Delta_t f_j(K) = \sum_{\substack{i_0, i_1 \geqslant 0 \\ i_0+i_1+j=d-1}} \binom{a_0}{i_0} \binom{a_1}{i_1} 2^{i_0} \quad (1 \leqslant j \leqslant d-1). \tag{50}$$

The corresponding contribution to the short cubical h-polynomial is therefore

$$\Delta_t h_K^{(\mathrm{sc})}(q) = \sum_{j=0}^{d-1} (\Delta_t f_j)(2q)^j (1-q)^{d-1-j} \tag{51}$$

$$= \sum_{\substack{i_0, i_1, j \geqslant 0 \\ i_0+i_1+j=d-1}} \binom{a_0}{i_0} \binom{a_1}{i_1} 2^{i_0} (2q)^j (1-q)^{d-1-j} \tag{52}$$

$$= \sum_{i_0, i_1 \geqslant 0} \binom{a_0}{i_0} \binom{a_1}{i_1} 2^{i_0} (2q)^{d-1-i_0-i_1} (1-q)^{i_0+i_1}. \tag{53}$$

Note that, in the last summation, each nonzero summand indeed satisfies

$$i_0 + i_1 \leqslant a_0 + a_1 \leqslant d-1. \tag{54}$$

We can therefore proceed to obtain

$$\Delta_t h_K^{(\mathrm{sc})}(q) = (2q)^{d-1} \cdot \sum_{i_0 \geqslant 0} \binom{a_0}{i_0} 2^{i_0} (2q)^{-i_0} (1-q)^{i_0} \cdot \sum_{i_1 \geqslant 0} \binom{a_1}{i_1} (2q)^{-i_1} (1-q)^{i_1} \tag{55}$$

$$= (2q)^{d-1} \cdot [1 + (1-q)/q]^{a_0} \cdot [1 + (1-q)/2q]^{a_1} \tag{56}$$

$$= 2^{d-1-a_1} q^{d-1-a_0-a_1} (1+q)^{a_1} = 2^{a_0} (1+q)^{a_1} (2q)^{a_2}. \tag{57}$$

This is exactly the formula claimed in part (iii) of Theorem 5. The corresponding formula for the contribution to the (long) cubical h-polynomial can be readily verified using definition (2), from which it follows that

$$q \cdot \Delta_t h_K^{(\mathrm{sc})}(q) = (1+q) \cdot \Delta_t h_K^{(\mathrm{c})}(q) - \Delta_t h_0^{(\mathrm{c})} - q^{d+1} \cdot \Delta_t h_d^{(\mathrm{c})}. \tag{58}$$

If $a_1 \geqslant 1$ then $\Delta_t h_K^{(sc)}(q)$ is divisible by $1+q$, by formula (57). Since $t \neq 1$ in this case, it follows from (3) that $\Delta_t h_0^{(c)} = 0$ and therefore also $\Delta_t h_d^{(c)} = 0$, so that

$$q \cdot \Delta_t h_K^{(sc)}(q) = (1+q) \cdot \Delta_t h_K^{(c)}(q). \tag{59}$$

The special cases with $a_1 = 0$ may be easily dealt with separately.

Note. Unlike the simplicial case, the number of occurences of each shelling type need not be the same for all the shellings of a given shellable cubical complex. In particular, these numbers are *not* determined by the (long or short) cubical h-vector. For example, the boundary complex of the 3-cube has a shelling with types

$$(2,0,0),\ (1,1,0),\ (0,2,0),\ (0,2,0),\ (0,1,1),\ (0,0,2) \tag{60}$$

and also a shelling with types

$$(2,0,0),\ (1,1,0),\ (1,1,0),\ (0,1,1),\ (0,1,1),\ (0,0,2). \tag{61}$$

The corresponding computations of the short cubical h-polynomial are

$$2^2 + 2(1+q) + (1+q)^2 + (1+q)^2 + (1+q)(2q) + (2q)^2 \tag{62}$$

$$= 2^2 + 2(1+q) + 2(1+q) + (1+q)(2q) + (1+q)(2q) + (2q)^2 \tag{63}$$

$$= 8 + 8q + 8q^2. \tag{64}$$

5. Remarks and open problems

If v is a vertex of a cubical $(d-1)$-complex K then the link (or vertex figure) K/v is a simplicial $(d-2)$-complex. A very interesting observation of Hetyei may be stated as follows.

Theorem 9 (Hetyei [7]). *The short cubical h-polynomial of a cubical $(d-1)$-complex K is equal to the sum, over all vertices v of K, of the simplicial h-polynomials of K/v:*

$$h_K^{(sc)}(q) = \sum_{v \in V} h_{K/v}^{(s)}(q). \tag{65}$$

Using some known properties of simplicial h-vectors, this observation can provide an alternative proof to part (i) of Theorem 5. Some other conclusions may be summarized as follows.

Corollary 10. *The short cubical h-vector is nonnegative for every locally Cohen–Macaulay (not necessarily shellable) cubical complex, and is unimodal for the boundary complex of a convex cubical polytope.*

Analogous properties (which imply, but are not implied by, the above assertions) may be expected for the (long) cubical h-vector.

Question 1. Is it true that

$$h_i^{(c)} \geqslant 0 \quad (0 \leqslant i \leqslant d) \tag{66}$$

for every Cohen–Macaulay cubical $(d-1)$-complex?

Question 2. Is it true that

$$h_{i-1}^{(c)} \leqslant h_i^{(c)} \quad (1 \leqslant i \leqslant d/2) \tag{67}$$

for the boundary complex of a convex cubical d-polytope?

Note. Part (iii) of Theorem 5 provides an affirmative answer to Question 1 for a *shellable* complex. Theorem 6 above provides an affirmative answer to Question 2 for $i = 1$. The relations between Question 2 and the lower bound conjecture for cubical polytopes [8] will be discussed elsewhere.

Finally, let us recall that a very general notion of h-vector for convex polytopes (in fact, for arbitrary Eulerian posets) has been introduced by Stanley [14]. It is defined by recursion on the poset elements, is symmetric (for Eulerian posets), and is unimodal for rational convex polytopes. For (the boundary complex of) the d-cube, a rather complicated explicit formula has been derived by I. Gessel [14, pp. 193–194]; it is quite different from the result in Lemma 2(iv) above. See also [5] for a discussion of the impact of shellability on this h-vector. It has been suggested by Gil Kalai that using the cubical (rather than the simplicial) h-vector as a basic model may lead to a version of the generalized h-vector which is unimodal for every triangle-free polytope.

Further research into the various notions of h-vector for cubical polytopes will doubtlessly solve some of the current puzzles, replacing them by even more challenging problems.

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